Geometric construction of the quantum Hall effect in all even dimensions

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Abstract
The quantum Hall effects in all even dimensions are uniformly constructed.
Contrary to some recent accounts in the literature, the existence of quantum Hall
effects (QHE) does not *crucially* depend on the existence of division algebras.
For QHE on flat space of even dimensions, both the Hamiltonians and the
ground-state wavefunctions for a single particle are explicitly described. This
explicit description immediately tells us that QHE on a higher even-dimensional
flat space shares common features such as incompressibility with QHE on a
plane.

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1. Introduction

Recently there has been a flurry on a generalization of the quantum Hall effect (QHE) to four-dimensional flat space [1–10], inspired by a paper [11] by Zhang and Hu. In view of its significance to condensed matter physics and to fundamental physics (see the conclusion and the introduction in [11]), it is worth the effort to make a closer examination of this generalization of the QHE.

In their search for QHE on four-space (i.e., $\mathbb{R}^4$), Zhang and Hu follow Haldane's approach to the QHE problem in [12]. There are two steps in this approach: (1) study the quantum mechanics problem of a single charged particle under the influence of a natural background magnetic field of strength I on the sphere of radius R and (2) map the sphere to the flat Euclidean space by standard stereographical mapping and then take the thermodynamic limit as both I and R go to infinity while keeping I/R^2 constant to recover QHE on the plane. Put it differently, QHE on the two-sphere devised by Haldane is just a family of spherical approximations to QHE on two-space (i.e., $\mathbb{R}^2$), and the strategy adopted in [11] is to generalize the spherical models of Haldane to dimension four (using an earlier work of Yang [13]) and then take the thermodynamic limit to obtain QHE on four-space.

As a matter of fact, the natural generalization of QHE on the two-sphere goes beyond dimension four. This observation was also independently made (but not carried out) by Fabinger in [9] from the point of view of fuzzy spheres [14].

It is well known that, to understand the QHE, a key step is to understand the Hamiltonian and the ground-state wavefunctions for a single particle on flat space. However, to our best understanding of the references cited above, this very important question for QHE on four-space has not been explicitly addressed so far. Our answer to this question for QHE on higher dimensional flat spaces turns out to be rather simple mathematically. This simplicity immediately tells us that QHE on higher dimensional flat space shares common features such as incompressibility with QHE on two-space.

2. QHE on even-dimensional spheres

Here we formulate the quantum mechanics model[1] for the QHE problem on even-spheres in clean geometric language. The approach in [11], where the ground-state wavefunctions are given first and the Hamiltonian is derived later, while it works in dimension four, does not seem to work in higher even dimensions. Our approach starts with the Hamiltonian and is more straightforward and works in any even dimension. However, the discussion here may not be really new in a broad sense, and it could have been known a long time ago to mathematicians in the context of representation theory of compact Lie groups. In any case, similar discussions in more general settings appeared (partially or fully) in the mathematical physics literature repeatedly in the past [16–18].

Following [11], a point X_i on $S^{2n}(R)$ (the $2n$ sphere centred at the origin with radius R) can be described by dimensionless vector coordinates $x_i = X_i/R$, with $i = 1, 2, \ldots, 2n + 1$, which satisfy $x_i^2 = 1$. Now, $S^{2n} \equiv S^{2n}(1)$ is the homogeneous space $\mathrm{spin}(2n + 1)/\mathrm{spin}(2n)$, and the principal bundle

$$\mathrm{spin}(2n) \rightarrow \mathrm{spin}(2n + 1) \rightarrow S^{2n} \tag{1}$$

has a canonical connection

$$A = Pr_{so(2n)}(g^{-1}dg) \tag{2}$$

where $g^{-1}dg$ is the Cartan–Maurer form on $\mathrm{spin}(2n + 1)$ and $Pr_{so(2n)}$ is the orthogonal projection onto the Lie algebra of $\mathrm{spin}(2n)$. Let Δ be the fundamental spin representation of $\mathrm{spin}(2n + 1)$, then $\Delta = \Delta^+ \oplus \Delta^-$ as representations of $\mathrm{spin}(2n)$, where $\Delta^\pm$ are the positive/negative spin representations of $\mathrm{spin}(2n)$. The highest weight state of $\Delta^\pm$ is $\left| \underbrace{\frac{1}{2} \cdots \frac{1}{2}}_{n-1} \frac{\pm 1}{2} \right\rangle$. Let I be a positive half integer, Δ_I^+ be the irreducible representation of $\mathrm{spin}(2n)$ with highest weight state $\left| \underbrace{I \cdots I}_{n} \right\rangle$. (In general Δ_I^+ is an irreducible component of the $2I$-fold symmetric tensor product of Δ^+.) Form the complex vector bundle ξ_I: $\mathrm{spin}(2n + 1) \times_{\mathrm{spin}(2n)} \Delta_I^+ \rightarrow S^{2n}$, then ξ_I has an induced canonical $\mathrm{spin}(2n)$-connection A_I.

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[1] Before going into the mathematical details, I would like to point out that, physically, it is the quantum mechanics of a charged particle in a $2n$-dimensional sphere under the influence of a canonical background $\mathrm{spin}(2n)$-gauge field. For further explanation, consult appendix A.

The quantum mechanics problem is the study of a charged particle in the presence of background magnetic potential A_I, so the Hamiltonian[2] is

$$\hat{H} = \frac{\hbar^2}{2MR^2} d_{A_I}^\dagger d_{A_I} \tag{3}$$

where d_{A_I} is the covariant derivative: $\Gamma(\xi_I) \to \Gamma(\xi_I \otimes T^*S^{2n})$, and $d_{A_I}^\dagger$ is the formal adjoint of d_{A_I}.

To compute the spectrum of $\hat{H}$ in equation (3), we note that

$$d_{A_I}^\dagger d_{A_I} = c_2\big(\text{spin}(2n+1)\big) - c_2\big(\text{spin}(2n), \Delta_I^+\big) \tag{4}$$

where $c_2(\text{spin}(2n+1))$ is the quadratic Casimir operator of $\text{spin}(2n+1)$ and $c_2\big(\text{spin}(2n), \Delta_I^+\big)$ is the value of the quadratic Casimir operator of $\text{spin}(2n)$ on Δ_I^+. Therefore we have

$$\hat{H} = \frac{\hbar^2}{2MR^2}\big[c_2(\text{spin}(2n+1)) - c_2\big(\text{spin}(2n), \Delta_I^+\big)\big]. \tag{5}$$

Note that equations (3) and (5) have appeared in [16–18] in a slightly different form (see appendix A for more details).

The Hilbert space of this quantum system is the space of square integrable sections of ξ_I and it decomposes into the direct sum of the eigenspaces of $\hat{H}$. These energy eigenspaces, indexed by integer $q \geqslant 0$, are all irreducible representation spaces of $\text{spin}(2n+1)$. The qth energy eigenspace $\mathcal{H}_I(q)$ is labelled by its highest weight state $\big|(q+I)\underbrace{I \cdots I}_{n-1}\big\rangle$, and the corresponding eigenvalue is $E(q) = \frac{\hbar^2}{2MR^2}\big[2I\big(q+\frac{n}{2}\big) + q(q+2n-1)\big]$. The ground state, which is the lowest $\text{spin}(2n+1)$ level for a given I, is obtained by setting $q = 0$, and is $d_0 \equiv \prod_{1\leqslant i \leqslant j \leqslant n}\big(1 + \frac{2I}{2n+1-i-j}\big)$-fold degenerate. (All these are standard results in mathematics and can be found in a textbook on group representations, for example [19].) Therefore, I plays the role of magnetic flux, while q plays the role of the Landau level index. States with $q > 0$ are separated from the ground state by a finite energy gap. Note that in the limit $I \to \infty$ and $R \to \infty$ while keeping $l_0 \equiv \frac{R}{\sqrt{2I}}$ and q constant,

$$E(q) \to \frac{\hbar^2}{2Ml_0^2}\big(q + \frac{n}{2}\big) \tag{6}$$

and the single particle energy spacing is finite. This can also be seen from the thermodynamic limit of $\hat{H}$ below.

3. QHE on even-dimensional flat spaces

Here we shall see that $\hat{H}$ in equation (3) is rather simple in the thermodynamic limit. As a first step, we shall find the expression for $\hat{H}$ on $S^{2n}\backslash\{S\}$ (i.e., the sphere with the south pole S removed). To do this, we need to fix a gauge on $S^{2n}\backslash\{S\}$, i.e., a smooth section ϕ on $S^{2n}\backslash\{S\}$ of the principal bundle in (1). We prefer to choose an $SO(2n)$-equivariant gauge, for example, we may take

$$\text{Landau gauge: } \phi(\vec{y}) = \begin{pmatrix} I - \frac{2\vec{y}\vec{y}^T}{1+y^2} & \frac{2\vec{y}}{1+y^2} \\ -\frac{2\vec{y}^T}{1+y^2} & \frac{1-y^2}{1+y^2} \end{pmatrix} \tag{7}$$

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[2] Here is a remark for QHE on a four-sphere. Note that $\text{spin}(4) = SU(2) \times SU(2)$, so the background gauge field splits into two components. However, since the particle is neutral with respect to the second component gauge field, physically, the particle only sees $SU(2)$—a component of $\text{spin}(4)$. So the effective principal bundle used here is just a $SU(2)$ bundle which can be seen to be precisely the Hopf bundle used in [11]. (These observations have already appeared in a series of papers by Y S Wu and his collaborators which were published in some Chinese journals in the 1970s.) The advantage of our construction is that the $\text{spin}(5)$ (not $SO(5)$) symmetry of the system is manifest from the very beginning. Compare with [11].

where $\vec{y}$ is the standard stereographical map $S^{2n} \setminus \{S\} \to \mathbb{R}^{2n}$, viewed as a coordinate map. Then the connection form on $S^{2n} \setminus \{S\}$ under the Landau gauge is

$$\phi^*(A) = 2(\vec{y}d\vec{y}^T - d\vec{y}\vec{y}^T) + O(|\vec{y}|^3) \tag{8}$$

where $O(|\vec{y}|^3)$ is a term whose coefficient in each dy^i is of order $|\vec{y}|^3$ as $\vec{y} \to \vec{0}$.

For the purpose of taking limits, it is convenient to have another representation of $SO(2n)$ [19]: a complex $2n \times 2n$ matrix X is in $SO(2n)$ if and only if it is anti-Hermitian and $JX^T J + X = 0$, where $J = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$ with I_n being the $n \times n$ identity matrix. In this representation, the natural connection form in equation (8) becomes

$$B = \begin{pmatrix} zdz^\dagger - dzz^\dagger & zdz^T - dzz^T \\ zdz^T - dzz^T & zdz^\dagger - dzz^\dagger \end{pmatrix} + O(|z|^3) \tag{9}$$

where $z = \begin{pmatrix} z^1 \\ \vdots \\ z^n \end{pmatrix}$ with $z^\mu = y^\mu + \sqrt{-1}y^{n+\mu}$ for $1 \leqslant \mu \leqslant n$ and a bar means complex conjugation. Then the Hamiltonian operator (in the Landau gauge) is

$$\hat{H} = -\frac{1}{2MR^2} \frac{1}{\sqrt{h}}(\partial_j + B_j)[h^{jk}\sqrt{h}(\partial_k + B_k)] \tag{10}$$

where h is the following Riemannian metric on $\mathbb{C}^n = \mathbb{R}^{2n}$:

$$h_{ij}(\vec{y}) = \frac{c}{1 + \vec{y}^2}\delta_{ij} \tag{11}$$

where c is a constant which can/will be set to be 1.

To find the thermodynamic limit of the Hamiltonian in equation (10), we write $B = B_\mu dz^\mu + B_{\bar{\mu}} d\bar{z}^\mu$, replace z by $z/\sqrt{2I}$, $\vec{y}$ by $\vec{y}/\sqrt{2I}$ and R^2 by $2I$ and observe that as $I \to \infty$,

$$B_\mu/\sqrt{2I} \to -\tfrac{1}{2}\bar{z}^\mu \qquad B_{\bar{\mu}}/\sqrt{2I} \to \tfrac{1}{2}z^\mu \tag{12}$$

as operators, and

$$h_{ij} \to \delta_{ij} \tag{13}$$

therefore,

$$\hat{H} \to \hat{H}_\infty = \sum_{1 \leqslant \mu \leqslant n} -\frac{1}{M}(\nabla_\mu \nabla_{\bar{\mu}} + \nabla_{\bar{\mu}} \nabla_\mu) \tag{14}$$

where $\nabla_\mu = \partial_\mu - \frac{1}{2}\bar{z}^\mu$ and $\nabla_{\bar{\mu}} = \partial_{\bar{\mu}} + \frac{1}{2}z^\mu$. In other words, the Hamiltonian for a single charged particle in QHE on $2n$ space is

$$\hat{H}_\infty = \sum_{1 \leqslant \mu \leqslant n} \left(-\nabla_\mu \nabla_{\bar{\mu}} + \frac{1}{2}\right)\hbar\omega_c. \tag{15}$$

Roughly speaking, equation (15) says that $\hat{H}_\infty$ is just the sum of n copies of the Hamiltonian for a single particle in QHE on two-space.

Main results[3] *for QHE on even-dimensional spaces.* Let $\hat{h}$ be the Hamiltonian of a single particle in QHE on two-space in the Landau gauge, V be the Bargmann–Fock space [15] of

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[3] Before taking the limit, the physics is about a charged particle on a sphere under the influence of a natural background gauge field. However, the limit does not have this kind of interpretation any longer. In the semiclassical picture, a $2n$-dimensional QHE droplet is a finite ball in the configuration space (not $\mathbb{R}^{2n}$) whose Landau levels up to the boundary are all filled. This is clear from our explicit description of the ground-state wavefunctions.

holomorphic functions on $\mathbb{C}$. Then the Hamiltonian of a single particle in QHE on four-space (in the Landau gauge) is

$$\hat{H}_\infty = \hat{h} \otimes I \otimes I + I \otimes \hat{h} \otimes I \tag{16}$$

and it acts on the Hilbert space $L^2(\mathbb{R}^2) \otimes L^2(\mathbb{R}^2) \otimes V$ (here I is the identity operator). The spectrum of this Hamiltonian is

$$E(q) = (q + 1)\hbar\omega_c \tag{17}$$

where $q = 0, 1, 2, \ldots$ and the Hilbert space of ground states is the Bargmann–Fock space of holomorphic functions on $\mathbb{C}^3$ with orthonormal basis (when we take the magnetic length as 1)

$$\psi_{\mathbf{k}}(z) = \frac{z^{[\mathbf{k}]}}{\sqrt{\pi^3 \mathbf{k}!}} \exp\left(-\frac{1}{2}|z|^2\right) \tag{18}$$

where $\mathbf{k} \equiv (k_1, k_2, k_3) \in \mathbb{Z}_+^3$, $|z|^2 \equiv |z_1|^2 + |z_2|^2 + |z_3|^2$, $z^{[\mathbf{k}]} \equiv z_1^{k_1} z_2^{k_2} z_3^{k_3}$ and $\mathbf{k}! \equiv k_1! k_2! k_3!$.

A similar conclusion also holds for QHE on $2n$ space with $n > 2$. In particular, the spectrum of the Hamiltonian is

$$E(q) = \left(q + \frac{n}{2}\right)\hbar\omega_c \tag{19}$$

where $q = 0, 1, 2, \ldots$ and the Hilbert space of ground states is the Bargmann–Fock space of holomorphic functions on $\mathbb{C}^{\frac{n(n+1)}{2}}$, the configuration space.

One may wonder whether the spectrum and the ground-state wavefunctions for a single particle in QHE on $2n$ space would be the same if they are obtained as the thermodynamic limit of their counterparts in QHE on $2n$ spheres. The answer is yes. The spectrum is seen to be the same by comparing equation (19) with equation (6), and the ground-state wavefunctions can be seen to be the same by doing a little further work (see appendix B).

4. Conclusion

The recent generalization of the QHE to four-space has been viewed as a significant attempt to answer fundamental questions in physics [11], and it has attracted considerable attention from physicists. In this paper, a clean geometric construction for this generalization is presented; and it actually yields a sequence of models of the QHE type:

QHE on 2 *space*, *Zhang–Hu model*, *QHE on* 6 *space*, *QHE on* 8 *space*[4], ...

Contrary to the accounts in [1, 11], the existence of quantum Hall effects does not *crucially* depend on the existence of division algebras.

Moreover, the Hamiltonian and the ground-state wavefunctions for a single particle on flat space are derived and explicitly described. The simplicity of this description immediately tells us that QHE on higher dimensional flat space shares common features such as incompressibility with QHE on two-space. We hope the simplicity of this description can also solve some mysteries surrounding QHE on higher dimensional spaces and thus facilitate the search for the grand unification based on QHE on four-space [11].

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[4] In dimension $8k$, if we use a real chiral spinor in our construction, the configuration space will be smaller, its dimension will be reduced from $8k + 4k(4k - 1) = 4k(4k + 1)$ to $8k + \frac{4k(4k - 1)}{2} = 2k(4k + 3)$, i.e., from 20 to 14 when $k = 1$. Compare with [1].

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Appendix A. Quantum mechanics of a charged particle on homogeneous spaces

The quantum mechanics of a charged particle on homogeneous spaces has been discussed in many recent papers in mathematical physics [17, 18] from more algebraic or computational points of view. Here we give a short presentation of it in geometric language. No originality is claimed because the construction is tautological and obvious to a modern geometer and has already appeared explicitly in [16] (at least) in the special case.

Appendix A.1. Generality

Let (X, h) be a Riemannian manifold with Riemannian metric h. Consider the quantum mechanics of a neutral particle of mass M freely moving in X; it is well known that the Hamiltonian operator is

$$\hat{H} = \frac{\hbar^2}{2M} \Delta \tag{A1}$$

where Δ is the semipositive definite Laplace operator. (In flat Euclidean space, $\Delta = -\sum_i \partial_i^2$.) The Hilbert space of this quantum mechanics problem is the space of square integrable complex-valued functions on X. Following Hodge, we write

$$\Delta = d^\dagger d \tag{A2}$$

where d is the exterior differential operator on complex-valued functions and $d^\dagger$ is its formal adjoint. In a local coordinate system, we have

$$\Delta = -\frac{1}{\sqrt{h}} \partial_i (h^{ij} \sqrt{h} \partial_j) \tag{A3}$$

where $h = \det(h_{ij})$.

Next we assume the particle is charged, and there is a background gauge field. Geometrically, a background gauge field is just a connection A on a certain Hermitian vector bundle ξ on X. A connection A is equivalent to a linear operator

$$d_A : \Gamma(\xi) \to \Gamma(T^*X \otimes \xi) \tag{A4}$$

satisfying the Leibnitz rule: $d_A(fs) = df \otimes s + f d_A s$, where s is a section of ξ and f is a function on X. The Riemannian metric on X together with the Hermitian metric on ξ makes possible the definition of the formal adjoint of d_A (denoted by $d_A^\dagger$). The obvious generalization of Δ in equation (2) is

$$\Delta_A = d_A^\dagger d_A. \tag{A5}$$

In a local coordinate system, with a choice of a gauge, we have

$$\Delta_A = -\frac{1}{\sqrt{h}} (\partial_j + A_j)(h^{jk} \sqrt{h} (\partial_k + A_k)) \tag{A6}$$

where $A_i dx^i$ is the Lie algebra-valued one-form representing the connection A in the fixed gauge.

The obvious generalization of equation (1) is

$$\hat{H} = \frac{\hbar^2}{2M} \Delta_A. \tag{A7}$$

For a general (X, h) and a general (ξ, A), the quantum mechanics problem is difficult to solve. However, for homogeneous space X and the associated canonical (ξ, A), the quantum mechanics problem is exactly soluble.

Appendix A.2. Quantum mechanics on homogeneous space

Let G be a reductive Lie group, and H be a compact Lie subgroup. The Cartan–Killing metric on G gives rise to a canonical Riemannian metric on the homogeneous space G/H. The principal H bundle

$$H \to G \to G/H \tag{A8}$$

has a canonical connection:

$$A(g) = Pr_{\mathfrak{h}}(g^{-1}dg) \tag{A9}$$

where $Pr_{\mathfrak{h}}$ is the orthogonal projection onto $\mathfrak{h}$ (the Lie algebra of H) using the Cartan–Killing metric on $\mathfrak{g}$ (the Lie algebra of G).

Let V be an irreducible unitary representation of H, forming the vector bundle

$$\xi_V : G \times_H V \to G/H \tag{A10}$$

The quantum mechanics problem discussed in a previous subsection, when applied in this setting, is completely soluble. It turns out that the problem can be fully described in terms of the representation theory of Lie groups: the Hilbert space, being the square integrable sections of ξ_V, is called the induced representation of G (induced from V), and the Hamiltonian is

$$\hat{H} = \frac{\hbar^2}{2M} (c_2(G) - c_2(H, V)) \tag{A11}$$

where $c_2(G)$ is the quadratic Casimir operator of G and $c_2(H, V)$ is the value of the quadratic Casimir operator of H on V. Here we have used the equation

$$d_A^\dagger d_A = c_2(G) - c_2(H, V). \tag{A12}$$

The proof of this equation is a simple exercise. The key observation is that both sides commute with the induced left action by G on a space of the H-equivariant map from G to V. (Note that a section of ξ_V is just an H-equivariant map from G to V.) Based on this observation, we just need to check that $d_A^* d_A \phi = (c_2(G) - c_2(H, V))\phi$ at the identity e of G. Next we choose an orthonormal basis $\{X_i\}$ at $T_e G$, such that the first p of the basis vectors forms an orthonormal basis for the orthogonal complement P of $T_e H$ in $T_e G$, so $P \cong \mathbb{R}^p$. Locally around eH, G/H is diffeomorphic to P under the exponential map, so it is also diffeomorphic to $\mathbb{R}^p$, and this defines a local coordinate map. The next observation is that these local coordinates are geodesic normal coordinates at eH. Then, using the definition of covariant derivatives, we have $d_A^* d_A \phi|_e = \sum_{i=1}^p X_i X_i \phi|_e = c_2(G)\phi|_e - c_2(H)\phi|_e$. The proof is completed by observing that $c_2(H)\phi|_e = c_2(H) \cdot \phi(e)$ (here $\cdot$ is the action of U$\mathfrak{h}$ on V).

Remark that the Hamiltonian discussed in [17, 18] is

$$\hat{H}' = c_2(G) \tag{A13}$$

and the one we use here appears explicitly in [16] in the case $G = SU(2)$ and $H = U(1)$.

Appendix B. Concrete description of wavefunctions of a single particle in QHE on even spheres

The main purpose here is to describe the wavefunctions of a single particle in QHE on even spheres at a fixed energy level as certain polynomials on the decompactified configuration space of QHE on even spheres.

For this purpose, we choose the Landau gauge on $S^{2n}\backslash\{S\}$ and identify $S^{2n}\backslash\{S\}$ with $\mathbb{C}^n$ under the standard stereographical projection map. Also, for each $k \geqslant 1$ we let Z_k denote the space of complex $k \times k$ matrices that are skew-symmetric about the second diagonal, and identify $\mathbb{C}^k \times Z_k$ with Z_{k+1} via

$$(z, z_k) \mapsto Z \equiv \begin{pmatrix} z & z_k \\ 0 & -\tilde{z} \end{pmatrix}. \tag{B1}$$

Here, for $z = (z_1, \ldots, z_k)^T$ (a column vector), we use $\tilde{z}$ to denote $(z_k, \ldots, z_1)$ (a row vector).

Note that Δ_I^+ has a concrete realization as the space of certain polynomials on Z_n [19]. Then a smooth section of ξ_I, being uniquely specified by a smooth map from $\mathbb{C}^n$ to Δ_I^+, can be realized as a smooth map from $\mathbb{C}^n \times Z_n$ to $\mathbb{C}$ or a smooth map from Z_{n+1} to $\mathbb{C}$.

Therefore, $\mathcal{H}_I(q)$ can be viewed as the space of certain smooth maps from Z_{n+1} to $\mathbb{C}$ in a natural way. On the other hand, $\mathcal{H}_I(q) \cong \Delta_I(q)$—the irreducible representation space of spin$(2n + 1)$ labelled by the highest weight state $\left|(q + I) \underbrace{I \cdots I}_{n-1}\right\rangle$. Since $\Delta_I(q)$ has a concrete realization as the space of certain polynomials on Z_{n+1} [19], the description of $\mathcal{H}_I(q)$ would be complete if we know the natural identification $\mathcal{H}_I(q) \cong \Delta_I(q)$. For this purpose, we define a self-diffeomorphism ϕ of Z_{n+1}: for $Z = \begin{pmatrix} z & z_n \\ 0 & -\tilde{z} \end{pmatrix}$, we have

$$\phi(Z) = \begin{pmatrix} z & (I_n + zz^\dagger)z_n(I_n + \tilde{z}^\dagger\tilde{z}) \\ 0 & -\tilde{z} \end{pmatrix} \tag{B2}$$

where I_n is the $n \times n$ identity matrix, and $\dagger$ means Hermitian conjugation. We are now ready to state our conclusion: the one-to-one correspondence $\mathcal{H}_I(q) \ni \psi_f \leftrightarrow f \in \Delta_I(q)$ is

$$\psi_f(Z) = f(\phi(Z)). \tag{B3}$$

Moreover, the integration measure used in defining the inner product on the space of wavefunctions is

$$d\mu_I = \frac{N_I(n) \, dZ}{(1 + |z|^2)^{2I+2n} \left|\det\left(I_n + z_n z_n^\dagger\right)\right|^{I+(n-1)}} \tag{B4}$$

where $N_I(n)$ is the normalization constant chosen such that $\int_{Z_{n+1}} d\mu_I = 1$.

Upon replacing Z by $Z/\sqrt{2I}$ and taking the limit $I \to \infty$, we have

$$d\mu_I \to d\mu_\infty = \pi^{-\frac{n(n+1)}{2}} e^{-|Z|^2} \, dZ \tag{B5}$$

i.e., the limit of the integration measure is the integration measure of Bargmann–Fock space [15]. It is then easy to see that as $I \to \infty$, $\Delta_I(0)$ approaches a subspace of the Bargmann–Fock space of holomorphic functions on $\mathbb{C}^{n(n+1)/2}$, and this subspace is the whole Bargmann–Fock space in the case $n = 1$ and $n = 2$. An induction argument actually shows that this subspace is the whole Bargmann–Fock space also in the general case.

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